13-3 Trigonometric Functions of General Angles

Main Ideas

- Find values of trigonometric functions for general angles.
- Use reference angles to find values of trigonometric functions.

New Vocabulary

quadrantal angle
reference angle

GET READY for the Lesson

A skycoaster consists of a large arch from which two steel cables hang and are attached to riders suited together in a harness. A third cable, coming from a larger tower behind the arch, is attached with a ripcord. Riders are hoisted to the top of the larger tower, pull the ripcord, and then plunge toward Earth. They swing through the arch, reaching speeds of more than 60 miles per hour. After the first several swings of a certain skycoaster, the angle θ of the riders from the center of the arch is given by $\theta = 0.2 \cos (1.6t)$, where t is the time in seconds after leaving the bottom of their swing.

Trigonometric Functions and General Angles In Lesson 13-1, you found values of trigonometric functions whose domains were the set of all acute angles, angles between 0 and $\frac{\pi}{2}$, of a right triangle. For $t > 0$ in the equation above, you must find the cosine of an angle greater than $\frac{\pi}{2}$. In this lesson, we will extend the domain of trigonometric functions to include angles of *any* measure.

KEY CONCEPT *Trigonometric Functions, θ in Standard Position*

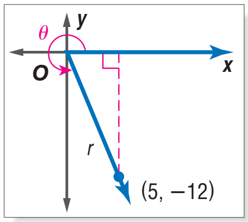
Let θ be an angle in standard position and let $P(x, y)$ be a point on the terminal side of θ. Using the Pythagorean Theorem, the distance r from the origin to P is given by $r = \sqrt{x^2 + y^2}$. The trigonometric functions of an angle in standard position may be defined as follows.

$\sin \theta = \frac{y}{r}$ $\quad\cos \theta = \frac{x}{r}$ $\quad\tan \theta = \frac{y}{x}, x \neq 0$

$\csc \theta = \frac{r}{y}, y \neq 0$ $\quad\sec \theta = \frac{r}{x}, x \neq 0$ $\quad\cot \theta = \frac{x}{y}, y \neq 0$

EXAMPLE 1 Evaluate Trigonometric Functions for a Given Point

Find the exact values of the six trigonometric functions of θ if the terminal side of θ contains the point (5, −12).



From the coordinates, you know that $x = 5$ and $y = -12$. Use the Pythagorean Theorem to find r.

$$r = \sqrt{x^2 + y^2} \quad \text{Pythagorean Theorem}$$
$$= \sqrt{5^2 + (-12)^2} \quad \text{Replace } x \text{ with 5 and } y \text{ with } -12.$$
$$= \sqrt{169} \text{ or } 13 \quad \text{Simplify.}$$

Now, use $x = 5$, $y = -12$, and $r = 13$ to write the ratios.

$$\sin \theta = \frac{y}{r} = \frac{-12}{13} \text{ or } -\frac{12}{13} \qquad \cos \theta = \frac{x}{r} = \frac{5}{13} \qquad \tan \theta = \frac{y}{x} = -\frac{12}{5} \text{ or } -\frac{12}{5}$$

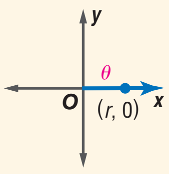
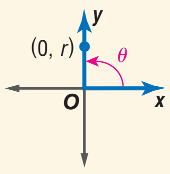
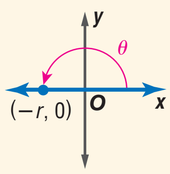
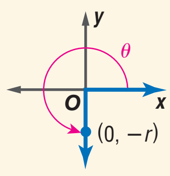
$$\csc \theta = \frac{r}{y} = \frac{13}{-12} \text{ or } -\frac{13}{12} \qquad \sec \theta = \frac{r}{x} = \frac{13}{5} \qquad \cot \theta = \frac{x}{y} = \frac{5}{-12} \text{ or } -\frac{5}{12}$$

CHECK Your Progress

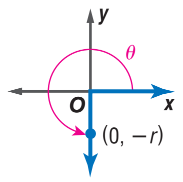
1. Find the exact values of the six trigonometric functions of θ if the terminal side of θ contains the point $(-8, -15)$.

If the terminal side of angle θ lies on one of the axes, θ is called a **quadrantal angle.** The quadrantal angles are 0°, 90°, 180°, and 270°. Notice that for these angles either x or y is equal to 0. Since division by zero is undefined, two of the trigonometric values are undefined for each quadrantal angle.

KEY CONCEPT — *Quadrantal Angles*

$\theta = 0°$ or 0 radians	$\theta = 90°$ or $\frac{\pi}{2}$ radians	$\theta = 180°$ or π radians	$\theta = 270°$ or $\frac{3\pi}{2}$ radians
			

EXAMPLE — Quadrantal Angles



2 **Find the values of the six trigonometric functions for an angle in standard position that measures 270°.**

When $\theta = 270°$, $x = 0$ and $y = -r$.

$$\sin \theta = \frac{y}{r} = \frac{-r}{r} \text{ or } -1 \qquad \cos \theta = \frac{x}{r} = \frac{0}{r} \text{ or } 0 \qquad \tan \theta = \frac{y}{x} = \frac{-r}{0} \text{ or undefined}$$

$$\csc \theta = \frac{r}{y} = \frac{r}{-r} \text{ or } -1 \qquad \sec \theta = \frac{r}{x} = \frac{r}{0} \text{ or undefined} \qquad \cot \theta = \frac{x}{y} = \frac{0}{-r} \text{ or } 0$$

CHECK Your Progress

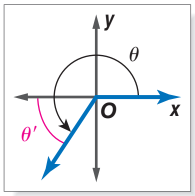
2. Find the values of the six trigonometric functions for an angle in standard position that measures 180°.

Reading Math

Theta Prime θ' is read *theta prime*.

Concepts in Motion
Animation
algebra2.com

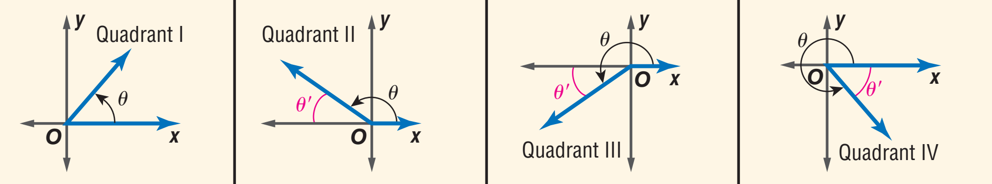
Reference Angles To find the values of trigonometric functions of angles greater than 90° (or less than 0°), you need to know how to find the measures of reference angles. If θ is a nonquadrantal angle in standard position, its **reference angle**, θ, is defined as the acute angle formed by the terminal side of θ and the x-axis.



You can use the rule below to find the reference angle for any nonquadrantal angle θ where $0° < \theta < 360°$ (or $0 < \theta < 2\pi$).

KEY CONCEPT — *Reference Angle Rule*

For any nonquadrantal angle θ, $0° < \theta < 360°$ (or $0 < \theta < 2\pi$), its reference angle θ' is defined as follows.



Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\theta' = \theta$	$\theta = 180° - \theta$ ($\theta' = \pi - \theta$)	$\theta' = \theta - 180°$ ($\theta' = \theta - \pi$)	$= 360° - \theta$ ($\theta' = 2\pi - \theta$)

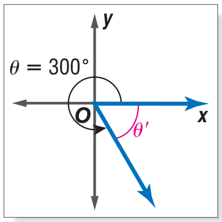
If the measure of θ is greater than 360° or less than 0°, its reference angle can be found by associating it with a coterminal angle of positive measure between 0° and 360°.

EXAMPLE **Find the Reference Angle for a Given Angle**

3 **Sketch each angle. Then find its reference angle.**

a. 300°

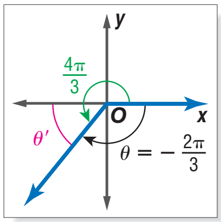
Because the terminal side of 300° lies in Quadrant IV, the reference angle is $360° - 300°$ or $60°$



b. $-\frac{2\pi}{3}$

A coterminal angle of $-\frac{2\pi}{3}$ is $2\pi - \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.

Because the terminal side of this angle lies in Quadrant III, the reference angle is $\frac{4\pi}{3} - \pi$ or $\frac{\pi}{3}$.



CHECK Your Progress

3A. $-200°$ **3B.** $\frac{2\pi}{3}$

To use the reference angle θ' to find a trigonometric value of θ, you need to know the sign of that function for an angle θ. From the function definitions, these signs are determined by x and y, since r is always positive. Thus, the sign of each trigonometric function is determined by the quadrant in which the terminal side of θ lies.

The chart summarizes the signs of the trigonometric functions for each quadrant.

	Quadrant			
Function	**I**	**II**	**III**	**IV**
$\sin\theta$ or $\csc\theta$	+	+	–	–
$\cos\theta$ or $\sec\theta$	+	–	–	+
$\tan\theta$ or $\cot\theta$	+	–	+	–

Use the following steps to find the value of a trigonometric function of any angle θ.

Step 1 Find the reference angle θ'.

Step 2 Find the value of the trigonometric function for θ'.

Step 3 Using the quadrant in which the terminal side of θ lies, determine the sign of the trigonometric function value of θ.

Study Tip

Look Back

To review **trigonometric values of angles measuring 30°, 45°, and 60°**, see Lesson 13-1.

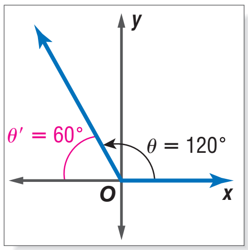
EXAMPLE Use a Reference Angle to Find a Trigonometric Value

4 **Find the exact value of each trigonometric function.**

a. $\sin 120°$

Because the terminal side of 120° lies in Quadrant II, the reference angle θ' is $180° - 120°$ or 60°. The sine function is positive in Quadrant II, so

$\sin 120° = \sin 60°$ or $\frac{\sqrt{3}}{2}$.



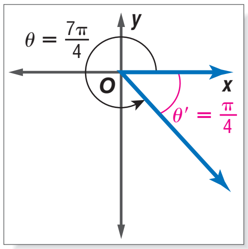
b. $\cot \frac{7\pi}{4}$

Because the terminal side of $\frac{7\pi}{4}$ lies in Quadrant IV, the reference angle θ' is $2\pi - \frac{7\pi}{4}$ or $\frac{\pi}{4}$. The cotangent function is negative in Quadrant IV.

$$\cot\frac{7\pi}{4} = -\cot\frac{\pi}{4}$$

$$= -\cot 45° \quad \frac{\pi}{4} \text{ radians } =5\ 45°$$

$$= -1 \quad \cot 45° =5\ 1$$



CHECK Your Progress

4A. $\cos 135°$

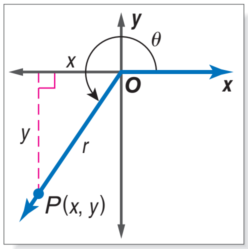
4B. $\tan\frac{5\pi}{6}$

If you know the quadrant that contains the terminal side of θ in standard position and the exact value of one trigonometric function of θ, you can find the values of the other trigonometric functions of θ using the function definitions.

EXAMPLE Quadrant and One Trigonometric Value of θ

5 **Suppose θ is an angle in standard position whose terminal side is in Quadrant III and $\sec \theta = -\frac{4}{3}$. Find the exact values of the remaining five trigonometric functions of θ.**

Draw a diagram of this angle, labeling a point $P(x, y)$ on the terminal side of θ. Use the definition of secant to find the values of x and r.



$\sec \theta = -\frac{4}{3}$ Given

$\frac{r}{x} = -\frac{4}{3}$ Definition of secant

Since x is negative in Quadrant III and r is always positive, $x = -3$ and $r = 4$. Use these values and the Pythagorean Theorem to find y.

$x^2 + y^2 = r^2$ Pythagorean Theorem

$(-3)^2 + y^2 = 4^2$ Replace x with -3 and r with 4.

$y^2 = 16 - 9$ Simplify. Then subtract 9 from each side.

$y = \pm\sqrt{7}$ Simplify. Then take the square root of each side.

$y = -\sqrt{7}$ y is negative in Quadrant III.

Use $x = -3$, $y = -\sqrt{7}$, and $r = 4$ to write the remaining trigonometric ratios.

$$\sin \theta = \frac{y}{r} = \frac{-\sqrt{7}}{4} \text{ or } -\frac{\sqrt{7}}{4}$$

$$\cos \theta = \frac{x}{r} = -\frac{3}{4}$$

$$\tan \theta = \frac{y}{x} = \frac{-\sqrt{7}}{-3} \text{ or } \frac{\sqrt{7}}{3}$$

$$\csc \theta = \frac{r}{y} = \frac{4}{-\sqrt{7}} \text{ or } -\frac{4\sqrt{7}}{7}$$

$$\cot \theta = \frac{x}{y} = \frac{-3}{-\sqrt{7}} \text{ or } \frac{3\sqrt{7}}{7}$$

CHECK Your Progress

5. Suppose θ is an angle in standard position whose terminal side is in Quadrant IV and $\tan \theta = -\frac{2}{3}$. Find the exact values of the remaining five trigonometric functions of θ.

Just as an exact point on the terminal side of an angle can be used to find trigonometric function values, trigonometric function values can be used to find the exact coordinates of a point on the terminal side of an angle.

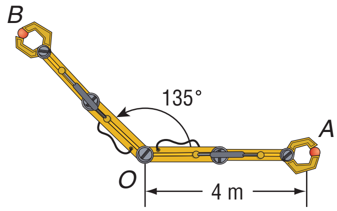
Real-World Link

RoboCup is an annual event in which teams from all over the world compete in a series of soccer matches in various classes according to the size and intellectual capacity of their robot. The robots are programmed to react to the ball and communicate with each other.

Source: www.robocup.corg

Real-World EXAMPLE Find Coordinates Given a Radius and an Angle

6 ROBOTICS In a robotics competition, a robotic arm 4 meters long is to pick up an object at point A and release it into a container at point B. The robot's arm is programmed to rotate through an angle of precisely 135° to accomplish this task. What is the new position of the object relative to the pivot point O?



With the pivot point at the origin and the angle through which the arm rotates in standard position, point A has coordinates (4, 0). The reference angle θ' for 135° is 180° – 135° or 45°.

Let the position of point B have coordinates (x, y). Then, use the definitions of sine and cosine to find the value of x and y. The value of r is the length of the robotic arm, 4 meters. Because B is in Quadrant II, the cosine of 135° is negative.

$\cos 135° = \frac{x}{r}$	cosine ratio
$-\cos 45° = \frac{x}{4}$	180° – 135° =5 45°
$-\frac{\sqrt{2}}{2} = \frac{x}{4}$	$\cos 45° = \frac{\sqrt{2}}{2}$
$-2\sqrt{2} = x$	Solve for x.

$\sin 135° = \frac{y}{r}$	sine ratio
$\sin 45° = \frac{y}{4}$	180° – 35° = 45°
$\frac{\sqrt{2}}{2} = \frac{y}{4}$	$\sin 45° = \frac{\sqrt{2}}{2}$
$2\sqrt{2} = y$	Solve for y.

The exact coordinates of B are $(-2\sqrt{2}, 2\sqrt{2})$. Since $2\sqrt{2}$ is about 2.83, the object is about 2.83 meters to the left of the pivot point and about 2.83 meters in front of the pivot point.

CHECK Your Progress

6. After releasing the object in the container at point B, the arm must rotate another 75°. What is the new position of the end of the arm relative to the pivot point O?

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CHECK Your Understanding

Example 1 (pp. 776–777)

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

1. (–15, 8) **2.** (–3, 0) **3.** (4, 4)

Examples 2, 4 (pp. 777, 779)

Find the exact value of each trigonometric function.

4. sin 300° **5.** cos 180° **6.** $\tan \frac{5\pi}{3}$ **7.** $\sec \frac{7\pi}{6}$

Example 3 (p. 778)

Sketch each angle. Then find its reference angle.

8. 235° **9.** $\frac{7\pi}{4}$ **10.** –240°

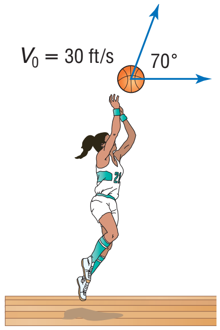
Example 5 (p. 780)

Suppose θ is an a ngle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

11. $\cos \theta = -\frac{1}{2}$, Quadrant II **12.** $\cot \theta = -\frac{\sqrt{2}}{2}$, Quadrant IV

Example 6 (p. 781)

13. BASKETBALL The maximum height H in feet that a basketball reaches after being shot is given by the formula $H = \frac{V_0^2 (\sin \theta)^2}{64}$, where V_0 represents the initial velocity and θ represents the degree measure of the angle that the path of the basketball makes with the ground. Find the maximum height reached by a ball shot with an initial velocity of 30 feet per second at an angle of 70°.



Exercises

HOMEWORK HELP	
For Exercises	See Examples
14–17	1
18–25	2, 4
26–29	3
30–33	5
34–36	6

Find the exact values of the six trigonometric functions of θ if the terminal side of θ in standard position contains the given point.

14. (7, 24) **15.** (2, 1) **16.** (5, −8) **17.** (4, −3)

18. (0, −6) **19.** (−1, 0) **20.** $(\sqrt{2}, -\sqrt{2})$ **21.** $(-\sqrt{3}, -\sqrt{6})$

Find the exact value of each trigonometric function.

22. sin 240° **23.** sec 120° **24.** tan 300° **25.** cot 510°

26. csc 5400° **27.** $\cos \frac{11\pi}{3}$ **28.** $\cot \left(-\frac{5\pi}{6}\right)$ **29.** $\sin \frac{3\pi}{4}$

30. $\sec \frac{3\pi}{2}$ **31.** $\csc \frac{17\pi}{6}$ **32.** cos (−30°) **33.** $\tan \left(-\frac{5\pi}{4}\right)$

Sketch each angle. Then find its reference angle.

34. 315° **35.** 240° **36.** $\frac{5\pi}{4}$ **37.** $\frac{5\pi}{6}$

38. −210° **39.** −125° **40.** $\frac{13\pi}{7}$ **41.** $-\frac{2\pi}{3}$

Suppose θ is an angle in standard position whose terminal side is in the given quadrant. For each function, find the exact values of the remaining five trigonometric functions of θ.

42. $\cos \theta = \frac{3}{5}$, Quadrant IV **43.** $\tan \theta = -\frac{1}{5}$, Quadrant II

44. $\sin \theta = \frac{1}{3}$, Quadrant II **45.** $\cot \theta = \frac{1}{2}$, Quadrant III

Real-World Link

If a major league pitcher throws a pitch at 95-miles per hour, it takes only about 4-tenths of a second for the ball to travel the 60-feet, 6-inches from the pitcher's mound to home plate. In that time, the hitter must decide whether to swing at the ball and if so, when to swing.

Source: exploratorium.edu

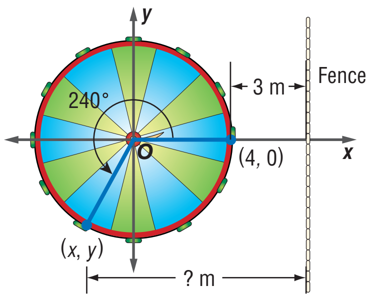
BASEBALL For Exercises 46 and 47, use the following information.

The formula $R = \frac{V_0^2 \sin 2\theta}{32}$ gives the distance of a baseball that is hit at an initial velocity of V_0 feet per second at an angle of θ with the ground.

46. If the ball was hit with an initial velocity of 80 feet per second at an angle of 30°, how far was it hit?

47. Which angle will result in the greatest distance? Explain your reasoning.

48. CAROUSELS Anthony's little brother gets on a carousel that is 8 meters in diameter. At the start of the ride, his brother is 3 meters from the fence to the ride. How far will his brother be from the fence after the carousel rotates 240°?



Otto Greule/Allsport

49. **SKYCOASTING** Mikhail and Anya visit a local amusement park to ride a skycoaster. After the first several swings, the angle the skycoaster makes with the vertical is modeled by $\theta = 0.2 \cos \pi t$, with θ measured in radians and t measured in seconds. Determine the measure of the angle for $t = 0$, 0.5, 1, 1.5, 2, 2.5, and 3 in both radians and degrees.

EXTRA PRACTICE
See pages 920, 938
Math Online
Self-Check Quiz at algebra2.com

50. **NAVIGATION** Ships and airplanes measure distance in nautical miles. The formula 1 nautical mile $= 6077 - 31 \cos 2\theta$ feet, where θ is the latitude in degrees, can be used to find the approximate length of a nautical mile at a certain latitude. Find the length of a nautical mile where the latitude is 60°.

H.O.T. Problems

51. **OPEN ENDED** Give an example of an angle for which the sine is negative and the tangent is positive.

52. **REASONING** Determine whether the following statement is *true* or *false*. If true, explain your reasoning. If false, give a counterexample.
The values of the secant and tangent functions for any quadrantal angle are undefined.

53. ***Writing in Math*** Use the information on page 776 to explain how you can model the position of riders on a skycoaster.

STANDARDIZED TEST PRACTICE

54. **ACT/SAT** If the cotangent of angle θ is 1, then the tangent of angle θ is

A –1. **C** 1.
B 0. **D** 3.

55. **REVIEW** Which angle has a tangent and cosine that are both negative?

F 110° **H** 210°
G 180° **J** 340°

Spiral Review

Rewrite each degree measure in radians and each radian measure in degrees. (Lesson 13–2)

56. 90°

57. $\frac{5\pi}{3}$

58. 5

59. **LITERATURE** In one of *Grimm's Fairy Tales*, Rumpelstiltskin has the ability to spin straw into gold. Suppose on the first day, he spun 5 pieces of straw into gold, and each day thereafter he spun twice as much. How many pieces of straw would he have spun into gold by the end of the week? (Lesson 11-4)

Use Cramer's Rule to solve each system of equations. (Lesson 4-6)

60. $3x - 4y = 13$
$-2x + 5y = -4$

61. $5x + 7y = 1$
$3x + 5y = 3$

62. $2x + 3y = -2$
$-6x + y = -34$

GET READY for the Next Lesson

PREREQUISITE SKILL **Solve each equation. Round to the nearest tenth.** (Lesson 13-1)

63. $\frac{a}{\sin 32°} = \frac{8}{\sin 65°}$

64. $\frac{b}{\sin 45°} = \frac{21}{\sin 100°}$

65. $\frac{c}{\sin 60°} = \frac{3}{\sin 75°}$